Completeness of propositional logic

In this subsection, we hope to convince you that the natural deduction rules of propositional logic are complete: whenever $\varphi 1, \varphi 2, \ldots, \varphi n \;\; \psi$ holds, then there exists a natural deduction proof for the sequent $\varphi 1, \varphi 2, \ldots, \varphi n \; \psi$. Combined with the soundness result of the previous subsection, we then obtain

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi \text{ is valid} \quad \text{iff} \quad \phi_1, \phi_2, \ldots, \phi_n \vDash \psi \text{ holds.}$$

This gives you a certain freedom regarding which method you prefer to use. Often it is much easier to show one of these two relationships (although neither of the two is universally better, or easier, to establish). The first method involves a proof search, upon which the logic programming paradigm is based. The second method typically forces you to compute a truth table which is exponential in the size of occurring propositional atoms. Both methods are intractable in general but particular instances of formulas often respond differently to treatment under these two methods.

The remainder of this section is concerned with an argument saying that if $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid. Assuming that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, the argument proceeds in three steps:

Step 1: We show that $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$ holds.
Step 2: We show that $\vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$ is valid.
Step 3: Finally, we show that $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.

The first and third steps are quite easy; all the real work is done in the second one.